PROJECTION OPERATOR METHOD FOR QUANTUM GROUPS

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1 Introduction

At present there can be no doubt to say that the quantum group theory has been one of the most important, modern and rapidly developing directions of mathematics and mathematical physics in the end of the twentieth century. Although initially (in the early 1980's) the quantum group theory was formulated for solving problems in the theory of integrable systems and statistical physics, later the surprising connection of this theory with many branches of mathematics, and the theoretical and mathematical physics was discovered. Today the quantum group theory is connected with such mathematical fields as special functions (especially, with q-orthogonal polynomials and basic hypergeometric series), the theory of difference and differential equations, combinatorial analysis and representation theory, matrix and operator algebras, noncommutative geometry, knot theory, topology, category theory and so on. From point of view of the mathematical physics there exists interconnection of the quantum group theory with the quantum inverse scattering method, conformal and quantum fields theory and so on. It is expected that the quantum groups will provide deeper understanding of concept of symmetry in physics.

In the broad sense the notation "quantum groups" [7] involves different deformations of the universal enveloping algebras $U(g)$ of Lie algebras and superalgebras g, such as: q-deformations, Yangians, elliptic and dynamic quantum groups, mixed deformations (for example, two-parameter deformations) and so on. In the restricted sense the quantum groups mainly mean the q-deformations of the universal enveloping algebras $U_q(g)$ of Lie algebras and superalgebras g (sometimes the q-deformations of the groups and supergroups are also included here). In what follows we use the notation of the quantum groups in this restricted sense.

It is well known that the method of projection operators for usual (non-quantized) Lie algebras and superalgebras is powerful and universal method for a solution of many problems in the representation theory. For example, the method allows to classify irreducible modules, to decompose modules on submodules (e.g. to analyze structure of Verma modules), to describe reduced (super)algebras (which are connected with reduction of a (super)algebra to (super)subalgebra), to construct bases of modules (e.g. the Gelfand-Tsetlin's type), to develop the detailed theory of Clebsch-Gordan coefficients and another elements

of Wigner-Racah calculus (including compact analytic formulas of these elements and their symmetry properties) and so on. It is evident that the projection operators of quantum groups play the same role in their representation theory.

In these lectures we develop the projection operator method for quantum groups. Here the term "quantum groups" means q-deformed universal enveloping algebras of contragredient Lie (super)algebras of finite growth (these (super)algebras include all finite-dimensional simple Lie algebras and classical superalgebras, infinite-dimensional affine (Kac-Moody) algebras and superalgebras). Conventionally, contains of the lectures can be divided on two parts. Basis fragments of the first part are: a combinatorial structure of root systems, the q-analog of the Cartan-Weyl basis, the extremal projector and the universal R-matrix for any contragredient Lie (super)algebra of finite growth. It should be noted that the explicit expressions for the extremal projectors and the universal R-matrices are ordered products of special q-series depending on noncommutative Cartan-Weyl generators.

In second part (Sects. 9-12) we consider some applications of the extremal projectors. Here we use the projector operator method to develop the theory of the Clebsch-Gordan coefficients for the quantum algebras $U_q(su(2))$ and $U_q(su(3))$. In particular, we give a very compact general formula for the canonical $U_q(su(3)) \supset U_q(su(2))$ Clebsch-Gordan coefficients in terms of the $U_q(su(2))$ Wigner $3nj$-symbols which are connected with the basic hyperheometric series. Then we apply the projection operator method for the construction of the q-analog of the Gelfand-Tsetlin basis for $U_q(su(n))$. Finally using analogy between the extremal projector $p(U_q(sl(2)))$ of the quantum algebra $U_q(sl(2))$ and the $\delta(x)$-function we introduce 'adjoint extremal projectors' $p^{(n)}(U_q(sl(2))$ $(n = 1, 2, \ldots)$ which are some generalizations of the extremal projector $p(U_q(sl(2)))$, and which are analogies of the derivatives of the $\delta(x)$-function, $\delta^{(n)}(x)$ $(n=1, 2, \ldots)$. The elements $p^{(n)}(U_q(sl(2))$ can be applied to construction and description of decomposable representations of quantum algebra $U_q(sl(2))$ (details see [6]).

2 Preliminary information

Let $g(A, \tau)$ be a contragredient Lie (super)algebra of finite growth[1] with a symmetrizable Cartan matrix A (i.e. $A = DA^{sym}$, where $A^{sym} = (a^{sym}_{ij})_{i,j\in I}$ is a symmetrical matrix, and D is an invertible diagonal matrix, $D = \mathrm{diag}\,(d_1, d_2, \ldots, d_r)$), $\tau \subset I$, $I := \{1, 2, \ldots, r\}$, and let $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ be a system of simple roots for $g(A, \tau)$.

The Lie (super)algebra $g := g(A, \tau)$ and its universal enveloping algebra $U(g)$ are completely determined by the Chevalley generators $e'_{\pm\alpha_i}$, h'_{α_i} $(i = 1, 2, \ldots, r)$ with the defining relations [9]:

$$[h'_{\alpha_i}, h'_{\alpha_j}] \;=\; 0\;, \qquad\qquad [h'_{\alpha_i}, e'_{\pm\alpha_j}] = \pm a^{sym}_{ij} e'_{\pm a_i}\;, \tag{2.1}$$

[1] These (super)algebras include all finite-dimensional simple Lie algebras and classical superalgebras, infinite-dimensional affine (Kac-Moody) algebras and superalgebras.

$$[e'_{\alpha_i}, e'_{-\alpha_j}] \; = \; \delta_{ij} h'_{\alpha_i}, \qquad (\mathrm{ad}\, e'_{\pm\alpha_i})^{n_{ij}}\, e'_{\pm\alpha_j} = 0 \quad \text{for} \;\; i \neq j \;, \tag{2.2}$$

where the positive integers n_{ij} are given as follows: $n_{ij} = 1$ if $a_{ii}^{sym} = a_{ij}^{sym} = 0$, $n_{ij} = 2$ if $a_{ii}^{sym} = 0$, $a_{ij}^{sym} \neq 0$, and $n_{ij} = -2a_{ij}^{sym}/a_{ii}^{sym} + 1$ if $a_{ii}^{sym} \neq 0$. Moreover it is necessary also to add all nontrivial relations of the form:

$$[[e'_{\pm\alpha_i}, e'_{\pm\alpha_j}], [e'_{\pm\alpha_i}, e'_{\pm\alpha_k}]] = 0 \tag{2.3}$$

for each triple of simple roots α_i, α_j, α_k if they satisfy the condition:

$$a_{ii}^{sym} = a_{jk}^{sym} = 0 \;, \qquad a_{ij}^{sym} = -a_{ik}^{sym} \neq 0 \;. \tag{2.4}$$

Throughout the paper the brackets $[\cdot, \cdot]$ and the symbol "ad" denote the supercommutator in $U(g)$, i.e.

$$(\mathrm{ad}\, a)b \equiv [a, b] = ab - (-1)^{\deg(a)\deg(b)} ba \tag{2.5}$$

for all homogeneous elements $a,\; b \in g$, where

$$\deg(h'_{\alpha_i}) = 0 \;\; \text{for}\; i \in I, \quad \deg(e'_{\pm\alpha_i}) = 0 \;\; \text{for}\; i \notin \tau, \quad \deg(e'_{\pm\alpha_i}) = 1 \;\; \text{for}\; i \in \tau. \tag{2.6}$$

Remarks. (i) The triple relation (2.3) may appear only in the supercase for the following situation in the Dynkin diagram:

$$\overset{\alpha_j}{\bullet} \!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \overset{\alpha_i}{\otimes} \!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \overset{\alpha_k}{\bullet} \tag{2.7}$$

Here α_i is an odd gray root, and α_j, a_k are not connected and they can be of any color (degree): white, gray or dark, and moreover the lines connected the node α_i with the nodes α_j and a_k can be non-single. The second equality (2.2) is ordinary called the Serre relation therefore the equality (2.3) may be called the triple Serre relation because it connects three root vectors e_{α_i}, e_{α_j} and e_{α_k}.
(ii) Besides the relations of type (2.3) the additional Serre relations of higher order can also occur but we don't give them here because these relations appear only for the Dynkin diagrams of special type. A total list of such diagrams and corresponding additional Serre relations can be found in the paper [30].

Let Δ_+ be a system of all positive roots of the (super)algebra $g(A, \tau)$. Any root γ of Δ_+ has the form: $\gamma = \sum_i^r n_i \alpha_i$, where all n_i are nonnegative integers. The total system of all roots, Δ, has the form: $\Delta = \Delta_+ \bigcup (-\Delta_+)$. On the system Δ there is a bilinear form $(\cdot, \cdot)$ such that $(\alpha_i, \alpha_j) = a_{ij}^{sym}$. The form is positive definite for all simple finite-dimensional Lie algebras and it is nondegenerate for all finite-dimensional contragredient Lie superalgebras. With respect to this form the simple roots $\alpha_i \in \Pi$ are classified (colored) as follows:

- A simple root α_i is called even (white) if $(\alpha_i, \alpha_i) \neq 0$ and $2\alpha_i \notin \Delta_+$. (In this case $i \notin \tau$).
- A simple root α_i is called odd, dark if $(\alpha_i, \alpha_i) \neq 0$ and $2\alpha_i \in \Delta_+$. (In this case $i \in \tau$).

- A simple root α_i is called odd, grey if $(\alpha_i, \alpha_i) = 0$. One can show that doubled grey roots don't exist, $2\alpha_i \notin \Delta_+$. (In this case $i \in \tau$).

The grey and dark roots occur only in the supercase.

Let γ be any root of Δ_+, this root is called odd if in its decomposition on the simple roots, $\gamma = \sum_i^r n_i\alpha_i$, the sum of the coefficients n_i for all odd roots α_i is odd. Otherwise the root γ is called even. The parity of the negative root $-\gamma$ coincides with the parity of the positive root γ.

Coloring of the roots is extended on all system Δ as follows:

- All even roots are white. A white root is pictured by the white node $\bigcirc$.
- An odd root γ is called grey if 2γ is not any root. This root is pictured by the grey node $\otimes$.
- An odd root γ is called dark if 2γ is a root. This root is pictured by the dark node $\bullet$.

In the case of the affine Kac-Moody (super)algebras all roots are also divided into real and imaginary. Every imaginary root $n\delta$ satisfies the condition $(n\delta, \gamma) = 0$ for all $\gamma \in \Delta$. For the real roots this condition is not valid.

3 Combinatorial structure of root systems

At first we remind the definition of the reduced system of the positive root system Δ_+ for any contragredient (super)algebras of finite growth.

Definition 3.1 *The system $\underline{\Delta}_+$ is called the reduced system if it is defined by the following way: $\underline{\Delta}_+ = \Delta_+ \backslash \{2\gamma \in \Delta_+ | \, \gamma \text{ is odd}\}$. That is the reduced system $\underline{\Delta}_+$ is obtained from the total system Δ_+ by removing of all doubled roots 2γ where γ is a dark odd root.*

Combinatorial structure of root system of the contragredient Lie (super)algebra of finite growth is connected with notation of the normal ordering in the reduced system of positive roots.

Definition 3.2 *We say that the system $\underline{\Delta}_+$ is in normal ordering if each composite (not simple) root $\gamma = \alpha + \beta$ $(\alpha, \beta, \gamma \in \underline{\Delta}_+)$, where α and β are not proportional roots $(\alpha \neq \lambda\beta)$, is written between its components α and β. It means that in the normal ordering system $\underline{\Delta}_+$ we have either*

$$\ldots, \alpha, \ldots, \alpha+\beta, \ldots, \beta, \ldots \, , \tag{3.1}$$

or

$$\ldots, \beta, \ldots, \alpha+\beta, \ldots, \alpha, \ldots \, . \tag{3.2}$$

We say also that $\alpha \prec \beta$ if α is located on the left side of β in the normal ordering system $\underline{\Delta}_+$, i.e. this corresponds to the case (3.1).

The normal ordering system $\underline{\Delta}_{+}$ is denoted by the symbol $\underline{\vec{\Delta}}_{+}$. It is evident that boundary (end) roots in $\underline{\vec{\Delta}}_{+}$ are simple. The combinatorial structure of the root system $\underline{\Delta}_{+}$ is described the following theorem.

Theorem 3.1 *(i) Normal ordering in the system $\underline{\Delta}_{+}$ exists for any mutual location of the simple roots $\alpha_i,\ i=1,2,\ldots,r$.*
(ii) Any two normal orderings $\underline{\vec{\Delta}}_{+}$ and $\underline{\vec{\Delta}}'_{+}$ can be obtained one from another by compositions of the following elementary inversions:

$$\alpha,\beta \leftrightarrow \beta,\alpha\ , \tag{3.3}$$

$$\alpha,\alpha{+}\beta,\beta \leftrightarrow \beta,\alpha{+}\beta,\alpha\ , \tag{3.4}$$

$$\alpha,\alpha{+}\beta,\alpha{+}2\beta,\beta \leftrightarrow \beta,\alpha{+}2\beta,\alpha{+}\beta,\alpha\ , \tag{3.5}$$

$$\alpha,\alpha{+}\beta,2\alpha{+}3\beta,\alpha{+}2\beta,\alpha{+}3\beta,\beta \leftrightarrow \beta,\alpha{+}3\beta,\alpha{+}2\beta,2\alpha{+}3\beta,\alpha{+}\beta,\alpha, \tag{3.6}$$

$$\begin{aligned}&\alpha,\delta{+}\alpha,2\delta{+}\alpha,\ldots,\infty\delta{+}\alpha,\delta,2\delta,3\delta,\ldots,\infty\delta,\infty\delta{-}\alpha,\ldots,2\delta{-}\alpha,\delta{-}\alpha \leftrightarrow\\ &\leftrightarrow\ \delta{-}\alpha,2\delta{-}\alpha,\ldots,\infty\delta{-}\alpha,\delta,2\delta,3\delta,\ldots,\infty\delta,\infty\delta{+}\alpha,\ldots,2\delta{+}\alpha,\delta{+}\alpha,\alpha\ ,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\alpha,\delta{+}\,2\alpha,\delta{+}\alpha,3\delta{+}2\alpha,2\delta{+}\alpha,\ldots,\infty\delta{+}\alpha,(2\infty{+}1)\delta{+}2\alpha,(\infty{+}1)\delta{+}\alpha,\delta,2\delta,\ldots,\\ &\infty\delta,(\infty{+}1)\delta{-}\alpha,(2\infty{+}1)\delta{-}2\alpha,\infty\delta-\alpha,\ldots,2\delta{-}\alpha,3\delta{-}2\alpha,\delta{-}\alpha,\delta{-}2\alpha, \leftrightarrow\\ &\leftrightarrow \delta{-}2\alpha,\delta{-}\alpha,3\delta{-}2\alpha,2\delta{-}\alpha,\ldots,\infty\delta{-}\alpha,(2\infty{+}1)\delta{-}2\alpha,(\infty{+}1)\delta{-}\alpha,\delta,2\delta,\ldots,\\ &\infty\delta,(\infty{+}1)\delta{+}\alpha,(2\infty{+}1)\delta{+}2\alpha,\infty\delta{+}\alpha,\ldots,2\delta{+}\alpha,3\delta{+}2\alpha,\delta{+}\alpha,\delta{+}2\alpha,\alpha,\end{aligned} \tag{3.8}$$

where $\alpha{-}\beta$ is not any root.

A proof of the second part *(ii)* for the case of the finite-dimensional simple Lie algebras can be found in [2]. The full proof of the theorem is given in the outgoing paper [28].

The root systems in (3.3)–(3.8) belong to the (super)algebras of rank 2. The combinatorial theorem permits to construct a q-analog of the Cartan-Weyl basis and to reduce the proof of basic theorems for extremal projector and the universal R-matrix for the quantum (super)algebra of arbitrary rank to the proof of such theorems for the quantum (super)algebras of rank 2.

4 Quantized Lie (super)algebras

The quantum (q-deformed) universal enveloping (super)algebra $U_q(g)$[2], where g is a contragredient Lie (super)algebra of finite growth, may be consider as a deformation f (reserving the grading) of the universal enveloping algebra $U(g)$: $U(g) \overset{f}{\mapsto} U_q(g)$ ($e'_{\pm\alpha_i} \overset{f}{\mapsto} e_{\pm\alpha_i}$, $h'_{\alpha_i} \overset{f}{\mapsto} h_{\alpha_i}$), which modifies the relations (2.2) (2.3). More precisely we have the following definition [26, 11].

[2]We shall also use the name "the quantized Lie (super)algebra g".

Definition 4.1 *The quantum (super)algebra $U_q(g)$ (where $g := g(A,\tau)$ is a contragredient Lie (super)algebra of finite growth), is an associative (super)algebra over $\mathbb{C}[q,q^{-1}]$ with Chevalley generators $e_{\pm\alpha_i}$, $k_{\alpha_i}^{\pm1} := q^{\pm h_{\alpha_i}}$, $(i \in I := \{1,2,\ldots,r\})$, and the defining relations:*

$$k_{\alpha_i}k_{\alpha_i}^{-1} = k_{\alpha_i}^{-1}k_{\alpha_i} = 1 \ , \qquad k_{\alpha_i}k_{\alpha_j} = k_{\alpha_j}k_{\alpha_i} \ , \tag{4.1}$$

$$k_{\alpha_i}e_{\pm\alpha_j}k_{\alpha_i}^{-1} = q^{\pm(\alpha_i,\alpha_j)}e_{\pm\alpha_j} \ , \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij}\frac{k_{\alpha_i}-k_{\alpha_i}^{-1}}{q-q^{-1}} \ , \tag{4.2}$$

$$(\mathrm{ad}_q\, e_{\pm\alpha_i})^{n_{ij}}e_{\pm\alpha_j} = 0 \qquad \text{for } \ i \neq j \ , \tag{4.3}$$

where the positive integers n_{ij} are the same as in the relations (2.2). Moreover, if any three simple roots α_i, α_j, α_k satisfy the condition (2.4) then there are the additional triple relations of the form:

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q, [e_{\pm\alpha_i}, e_{\pm\alpha_k}]_q]_q = 0 \ . \tag{4.4}$$

Here in (4.1)–(4.4) the brackets $[\cdot,\cdot]$ is the usual supercommutator (2.5), and $[\cdot,\cdot]_q$ and ad_q denote the q-deformed supercommutator (q-supercommutator) in $U_q(g)$:

$$(\mathrm{ad}_q\, e_\alpha)e_\beta \equiv [e_\alpha, e_\beta]_q = e_\alpha e_\beta - (-1)^{\deg(e_\alpha)\deg(e_\beta)}q^{(\alpha,\beta)}e_\beta e_\alpha \ , \tag{4.5}$$

where (α,β) is a scalar product of the roots α and β, and the parity function $\deg(\cdot)$ is given by

$$\deg(k_{\alpha_i}){=}0 \ \text{ for } i \in I, \quad \deg(e_{\pm\alpha_i}){=}0 \ \text{ for } i \notin \tau, \quad \deg(e_{\pm\alpha_i}){=}1 \ \text{ for } i \in \tau. \tag{4.6}$$

Below we shall use the following short notation:

$$\vartheta(\gamma) := \vartheta(e_\gamma) = \deg(e_\gamma) \ . \tag{4.7}$$

Remarks. (i) It is not hard to verify that the the relations (4.1)–(4.4) are invariant with respect to the replacement of q by q^{-1}.
(ii) The outer q-supercommutator in (4.4) is really the usual supercommutator since $(\alpha_i{+}\alpha_j, \alpha_i{+}\alpha_k){=}0$.
(iii) The remark (ii) after the formula (2.7) is also valid.

Clearly, the quantum (super)algebra $U_q(g)$ reduces to the usual universal enveloping (super)algebra $U(g)$ if $q \to 1$.

By direct calculations we can show that quantum (super)algebra $U_q(g)$ is a Hopf (super)algebra with respect to a comultiplication Δ_q, an antipode S_q and a counit ε defined as

$$\begin{aligned}
\Delta_q(k_{\alpha_i}^{\pm1}) &= k_{\alpha_i}^{\pm1}\otimes k_{\alpha_i}^{\pm1} \ , & S_q(k_{\alpha_i}^{\pm1}) &= k_{\alpha_i}^{\mp1} \ , \\
\Delta_q(e_{\alpha_i}) &= e_{\alpha_i}\otimes 1 + k_{\alpha_i}^{-1}\otimes e_{\alpha_i} \ , & S_q(e_{\alpha_i}) &= -k_{\alpha_i}e_{\alpha_i} \ , \\
\Delta_q(e_{-\alpha_i}) &= e_{-\alpha_i}\otimes k_{\alpha_i} + 1\otimes e_{-\alpha_i} \ , & S_q(e_{-\alpha_i}) &= -e_{-\alpha_i}k_{\alpha_i}^{-1} \ ,
\end{aligned} \tag{4.8}$$

$$\varepsilon(e_{\pm\alpha_i}) = 0 \ , \qquad \varepsilon(k_{\alpha_i}) = \varepsilon(1) = 1 \ , \tag{4.9}$$

Both in the quantum and non-quantum case we can directly use the Chevalley generators for construction of a 'monomial' basis in all universal enveloping (super)algebra $U_q(g)$ ($U(g)$). Bases of such kind were proposed by Verma for the non-quantized case and by Lusztig [17] for the general case. The Lusztig basis is an universal one and it is called the canonical basis. Both the Verma basis and the Lusztig basis have rather complicated algebraic structure and therefore they were not used in a broad fashion until now. It is well known that a monomial basis constructed of Cartan-Weyl generators is a more algebraically simple basis. Therefore a natural problem is to construct a q-analog of the Cartan-Weyl basis (the quantum Cartan-Weyl basis) for the quantum (super)algebra $U_q(g)$.

Our method for construction of the q-analog of the Cartan-Weyl basis and its general properties and also properties of the extremal projector and the universal R-matrix are closely connected with the combinatorial structure for the root system of the Lie (super)algebra g.

5 Quantum Cartan-Weyl basis

The q-analog of the Cartan-Weyl basis for $U_q(g)$ is constructed by using the following inductive algorithm [26], [11]–[15].
We fix some normal ordering $\vec{\underline{\Delta}}_+$ and put by induction

$$e_\gamma := [e_\alpha, e_\beta]_q, \qquad e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_{q^{-1}} \tag{5.1}$$

if $\gamma=\alpha+\beta$, $\alpha \prec \gamma \prec \beta$ ($\alpha,\beta,\gamma \in \underline{\Delta}_+$), and the segment $[\alpha;\beta] \subseteq \vec{\underline{\Delta}}_+$ is minimal one including the root γ, i.e. the segment has not another roots α' and β' such that $\alpha'+\beta'=\gamma$. Moreover we put

$$k_\gamma := \prod_{i=1}^{r} k_{\alpha_i}^{l_i} \ , \tag{5.2}$$

if $\gamma = \sum_{i=1}^{r} l_i\alpha_i$ ($\gamma \in \underline{\Delta}_+,\ \alpha_i \in \Pi$).
By this procedure one can construct the total quantum Cartan-Weyl basis for all quantized finite-dimensional simple contragredient Lie (super)algebras. In the case of the quantized infinite-dimensional affine Kac-Moody (super)algebras we have to apply one more additional condition. Namely, first we construct all root vectors e_γ ($\gamma \in \underline{\Delta}$) by means of the given procedure, and then we overdeterminate the generators $e_{n\delta}$ of the imaginary roots $n\delta \in \underline{\Delta}$ in a way that the new generators $e'_{n\delta}$ are mutually commutative if they are not conjugate generators. Because of the fact that we do not have a sufficient place here to describe the overdetermination of imaginary root generators in details, we are restricted to a consideration of finite-dimensional case, i.e. when g is a finite-dimensional simple contragredient Lie (super)algebra.

The quantum Cartan-Weyl basis is characterized by the following properties [11]–[15].

Proposition 5.1 *The root vectors $\{e_{\pm\gamma}\}$ ($\gamma \in \underline{\Delta}_+$) satisfy the following relations:*

$$k_\alpha^{\pm 1} e_\gamma = q^{\pm(\alpha,\gamma)} e_\gamma k_\alpha^{\pm 1} , \tag{5.3}$$

$$[e_\gamma, e_{-\gamma}] = a(\gamma)\frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}} , \tag{5.4}$$

$$[e_\alpha, e_\beta]_q = \sum_{\alpha\prec\gamma_1\prec\ldots\prec\gamma_n\prec\beta} C_{m_i,\gamma_i} e_{\gamma_1}^{m_1} e_{\gamma_2}^{m_2} \cdots e_{\gamma_n}^{m_n}, \tag{5.5}$$

where $\sum_i^n m_i\gamma_i = \alpha+\beta$, and the coefficients $C_{\ldots}$ are rational functions of q and they do not depend on the Cartan elements k_{α_i}, $i = 1, 2, \ldots n$, and also

$$[e_\beta, e_{-\alpha}] = \sum C'_{m_i,\gamma_i;m'_j,\gamma'_j} e_{-\gamma_1}^{m_1} e_{-\gamma_2}^{m_2} \cdots e_{-\gamma_p}^{m_p} e_{\gamma'_1}^{m'_1} e_{\gamma'_2}^{m'_2} \cdots e_{\gamma'_s}^{m'_s} \tag{5.6}$$

where the sum is taken on $\gamma_1, \ldots, \gamma_p, \gamma'_1, \ldots, \gamma'_s$ and $m_1, \ldots, m_p, m'_1, \ldots, m'_s$ such that

$$\gamma_1 \prec \ldots \prec \gamma_p \prec \alpha \prec \beta \prec \gamma'_1 \prec \ldots \prec \gamma'_s \ , \quad \sum_l (m'_l\gamma'_l - m_l\gamma_l) = \beta - \alpha$$

and the coefficients $C'_{\ldots}$ are rational functions of q and k_α or k_β. The monomials $e_{\gamma_1}^{n_1} e_{\gamma_2}^{n_2} \cdots e_{\gamma_p}^{n_p}$ and $e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_p}^{n_p}$, ($\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_p$), generate (as a linear space over $U_q(\mathcal{H})$) subalgebras $U_q(b_+)$ and $U_q(b_-)$ correspondingly. The monomials

$$e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_p}^{n_p} e_{\gamma'_1}^{n'_1} e_{\gamma'_2}^{n'_2} \cdots e_{\gamma'_s}^{n'_s},$$

where $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_p$ and $\gamma'_1 \prec \gamma'_2 \prec \cdots \prec \gamma'_s$), generate $U_q(g)$ over $U_q(\mathcal{H})$.

Here the algebra $U_q(\mathcal{H})$ is generated by the Cartan elements $k_{\alpha_i} (i=1, 2, \ldots, r)$.

Now we consider some extensions of $U_q(g)$, $U_q(b_+) \otimes U_q(b_-)$ and $U_q(g) \otimes U_q(g)$ since the extremal projector and the universal R-matrix are elements of these extensions.

6 Taylor extensions of $U_q(g)$, $U_q(b_+) \otimes U_q(b_-)$ and $U_q(g) \otimes U_q(g)$

Let Fract$(U_q(K))$ be a field of fractions over $U_q(K)$, i.e. Fract$(U_q(K))$ is an associative algebra of rational functions of the elements $k_{\alpha_i}^{\pm 1}$, ($i = 1, 2, \ldots, r$). We put

$$\tilde{U}_q(g) = \mathrm{Fract}\,(U_q(K)) \otimes_{U_q(K)} U_q(g) \ . \tag{6.1}$$

Evidently, the extension $\tilde{U}_q(g)$ is an associative algebra. The algebra $\tilde{U}_q(g)$ is called the Cartan extension of the quantum algebra $U_q(g)$.

Let $\{e_{\pm\gamma}\}$, $\gamma \in \underline{\Delta}_+$, be the root vectors of the quantum Cartan-Weyl basis built in accordance with some fixed normal ordering in $\underline{\Delta}_+$. Let us construct a formal Taylor series on the following monomials

$$e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{n_\alpha}\, e_{\alpha}^{m_\alpha} e_{\gamma}^{m_\gamma} \cdots e_{\beta}^{m_\beta} \tag{6.2}$$

with coefficients from Fract $(U_q(K))$, where $\alpha \prec \gamma \prec \cdots \prec \beta$ in a sense of the fixed normal ordering in $\underline{\Delta}_+$ and nonnegative integers $n_\beta, n_\gamma \ldots, n_\alpha, m_\alpha, m_\gamma \ldots, m_\beta$ are subjected to the constraints

$$\Big| \sum_{\gamma \in \underline{\Delta}_+} (n_\gamma - m_\gamma) c_i^{(\gamma)} \Big| \leq \text{const} , \qquad i = 1, 2, \cdots, r , \tag{6.3}$$

where $c_i^{(\gamma)}$ are coefficients in a decomposition of the root γ with respect to the system of simple roots Π. Let $T_q(g)$ be a linear space of all such formal series. We have the following simple proposition.

Proposition 6.1 *The linear space $T_q(g)$ is an associative algebra with respect to a multiplication of formal series.*

The algebra $T_q(g)$ is called the Taylor extension of $U_q(g)$.

Let Fract $(U_q(K \otimes K))$ be a field of fractions generated by the following elements: $1 \otimes k_{\alpha_i}$, $k_{\alpha_i} \otimes 1$ and $q^{h_{\alpha_i} \otimes h_{\alpha_j}}$, $(i, j = 1, 2, \ldots, r)$. Let us consider a formal Taylor series of the following monomials

$$e_{\alpha}^{n_\alpha} e_{\gamma}^{n_\gamma} \cdots e_{\beta}^{n_\beta} \otimes e_{-\beta}^{m_\beta} \cdots e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha} \tag{6.4}$$

with coefficients from Fract $(U_q(K \otimes K))$, where $\alpha \prec \gamma \prec \cdots \prec \beta$ in a sense of the fixed normal ordering in $\underline{\Delta}_+$ and nonnegative integers $n_\beta, \ldots, n_\alpha, m_\alpha, \ldots, m_\beta$ are subjected to the constraint (6.3). Let $T_q(b_+ \otimes b_-)$ be a linear space of all such formal series. The following proposition holds.

Proposition 6.2 *The linear space $T_q(b_+ \otimes b_-)$ is an associative algebra with respect to a multiplication of formal series.*

The algebra $T_q(b_+ \otimes b_-)$ will be called the Taylor extension of $U_q(b_+) \otimes U_q(b_-)$.

At least we consider a formal Taylor series of the following monomials

$$e_{-\beta}^{m_\beta} \cdots e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha} e_{\alpha}^{n_\alpha} e_{\gamma}^{n_\gamma} \cdots e_{\beta}^{n_\beta} \otimes e_{-\beta}^{m'_\beta} \cdots e_{-\gamma}^{m'_\gamma} e_{-\alpha}^{m'_\alpha} e_{\alpha}^{n'_\alpha} e_{\gamma}^{n'_\gamma} \cdots e_{\beta}^{n'_\beta} \tag{6.5}$$

with coefficients from Fract $(U_q(K \otimes K))$, where $\alpha \prec \gamma \prec \cdots \prec \beta$ in a sense of the fixed normal ordering in Δ_+ and nonnegative integers $n_\beta, \ldots, n_\alpha, m_\alpha, \ldots, m_\beta$ and $n'_\beta, \ldots, n'_\alpha, m'_\alpha, \ldots, m'_\beta$ are subjected to the constraints

$$\Big| \sum_{\gamma \in \Delta_+} (n_\gamma + n'_\gamma - m_\gamma - m'_\gamma) c_i^{(\gamma)} \Big| \leq \text{const} , \qquad i = 1, 2, \cdots, r . \tag{6.6}$$

Let $T_q(g \otimes g)$ be a linear space of all such formal series. The following simple proposition holds.

Proposition 6.3 *The linear space $T_q(g\otimes g)$ is an associative algebra with respect to a multiplication of formal series.*

The algebra $T_q(g\otimes g)$ will be called the Taylor extension of $U_q(g)\otimes U_q(g)$. Evidently the following embedding hold

$$\begin{aligned} T_q(g\otimes g) \supset T_q(b_+\otimes b_-)\ , \\ T_q(g\otimes g)\supset T_q(g)\otimes T_q(g)\supset \Delta_q(T_q(g))\ . \end{aligned} \tag{6.7}$$

7 Extremal projector

By definition, the extremal projector for $U_q(g)$ is a nonzero element $p := p(U_q(g))$ of the Taylor extension $T_q(g)$, satisfying the equations

$$e_{\alpha_i}p = pe_{-\alpha_i} = 0 \quad (\forall\ \alpha_i\in\Pi)\ , \qquad p^2 = p\ . \tag{7.1}$$

Acting by the extremal projector p on any highest weight $U_q(g)$-module M we obtain a space $M^0 = pM$ of highest weight vectors for M (if pM has no singularities).

Fix some normal ordering $\vec{\underline{\Delta}}_+$ and let $\{e_{\pm\gamma}\}$ ($\gamma\in\underline{\Delta}_+$) be the corresponding Cartan-Weyl generators. The following statement holds for any quantized finite-dimensional contragredient Lie (super)algebra[3] g [26, 12].

Theorem 7.1 *The equations (7.1) have a unique nonzero solution in the space of the Taylor extension $T_q(g)$ and this solution has the form*

$$p = \prod_{\gamma\in\vec{\underline{\Delta}}_+} p_\gamma\ , \tag{7.2}$$

where the order in the product coincides with the chosen normal ordering of $\underline{\Delta}_+$ and the elements p_γ are defined by the formulae

$$p_\gamma = \sum_{m\geq 0}\frac{(-1)^m}{(m)_{\bar{q}_\gamma}!}\,\varphi_{\gamma,m}e_{-\gamma}^m e_\gamma^m\ , \tag{7.3}$$

$$\varphi_{\gamma,m} \;=\; \frac{(q-q^{-1})^m q^{-\frac{1}{4}m(m-3)(\gamma,\gamma)}q^{-m(\rho,\gamma)}}{(a(\gamma))^m\prod\limits_{l=1}^{m}\Bigl(k_\gamma q^{(\rho,\gamma)+\frac{l}{2}(\gamma,\gamma)}-(-1)^{(l-1)\theta(\gamma)}k_\gamma^{-1}q^{-(\rho,\gamma)-\frac{l}{2}(\gamma,\gamma)}\Bigr)}\ . \tag{7.4}$$

Here ρ is a linear function such that $(\rho,\alpha_i) = \frac{1}{2}(\alpha_i,\alpha_i)$ for all simple roots $\alpha_i\in\Pi$; $a(\gamma)$ is a factor in the relation (5.4); $\bar{q}_\gamma := (-1)^{\theta(\gamma)}q^{-(\gamma,\gamma)}$; the symbol $(m)_q$ is given by the formula:

$$(m)_q := \frac{q^m-1}{q-1}\ . \tag{7.5}$$

[3]The theorem is also valid for the quantized infinite-dimensional affine Kac-Moody (super)algebras, but in this case the formulas (7.3) and (7.4) for the imaginary roots $\gamma = n\delta$ should be more detailed (see [10, 28] as examples).

In the limit $q\to 1$ we obtain the extremal projector for the (super)algebra g: $\lim\limits_{q\to 1} p(U_q(g)) = p(g)$ [1, 2, 24, 25]. A proof of the theorem actually reduces to the proof for the case of the quantized (super)algebras of rank 2, and it is similar to the case of non-deformed finite-dimensional simple Lie algebras [2].

8 Universal R-matrix

By definition, the universal R-matrix for the Hopf (super)algebra $U_q(g)$ is an invertible element of the Taylor extension $T_q(b_+\otimes b_-)$, satisfying the equations

$$\tilde{\Delta}_q(x) = R\Delta(x)R^{-1}, \qquad \forall\, x\in U_q(g), \tag{8.1}$$

$$(\Delta_q\otimes \mathrm{id})R = R^{13}R^{23}\ , \qquad (\mathrm{id}\otimes\Delta_q)R = R^{13}R^{12}, \tag{8.2}$$

where $\tilde{\Delta}_q$ is an opposite comultiplication: $\tilde{\Delta}_q = \sigma\Delta_q$, $\sigma(x\otimes y) = (-1)^{\deg x\deg y}y\otimes x$ for all homogeneous elements $x, y\in U_q(g)$. In (8.2) we use standard notation $R^{12} = \sum a_i\otimes b_i\otimes 1$, $R^{13} = \sum a_i\otimes 1\otimes b_i$, $R^{23} = \sum 1\otimes a_i\otimes b_i$ if R has a form $R = \sum a_i\otimes b_i$.
We employ the following standard notation for the q-exponential:

$$\exp_q(x) := 1 + x + \frac{x^2}{(2)_q!} + \ldots + \frac{x^n}{(n)_q!} + \ldots = \sum_{n\geq 0}\frac{x^n}{(n)_q!}\ , \tag{8.3}$$

where $(n)_q$ is defined by the formulas (7.5).

Fix some normal ordering $\underline{\vec{\Delta}}_+$ and let $\{e_{\pm\gamma}\}$ ($\gamma\in\underline{\Delta}_+$) be the corresponding Cartan-Weyl generators. The following statement holds for any quantized finite-dimensional contragredient Lie (super)algebra[4] g [11, 12, 13].

Theorem 8.1 *The equation (8.1) has a unique (up to a multiplicative constant) invertible solution in the space of the Taylor extension $T_q(b_+\otimes b_-)$ and this solution has the form*

$$R = \Big(\prod_{\gamma\in\underline{\vec{\Delta}}_+} R_\gamma\Big)\cdot K, \tag{8.4}$$

where the order in the product coincides with the chosen normal ordering $\underline{\vec{\Delta}}_+$ and the elements R_γ and K are defined by the formulas

$$R_\gamma = \exp_{\bar{q}_\gamma}\left((-1)^{\theta(\gamma)}(q-q^{-1})(a(\gamma))^{-1}(e_\gamma\otimes e_{-\gamma})\right), \tag{8.5}$$

$$K = q^{\sum_{i,j} d_{ij}(h_{\alpha_i}\otimes h_{\alpha_j})} \tag{8.6}$$

where $a(\gamma)$ is a factor from the relation (5.4), and d_{ij} is an inverse matrix for a symmetrical Cartan matrix (a_{ij}^{sym}) if (a_{ij}^{sym}) is not degenerated. (In a case of a

[4] The theorem is also valid for the quantized infinite-dimensional affine Kac-Moody (super)algebras, but in this case the formula (8.5) for the imaginary roots $\gamma = n\delta$ should be more detailed (see [29, 14, 15].

degenerated (a_{ij}^{sym}) we extend it up to a non-degenerated matrix ($\tilde{a}_{ij}^{sym}$) and take an inverse to this extended matrix). Moreover the solution (8.4) is the universal R-matrix, i.e. it satisfies the equations (8.2) too.

A proof of the theorem actually reduces to the proof for the case of the (super)algebras of rank 2 (see [11]).

In the rest sections we consider some applications of the extremal projectors.

9 Clebsch-Gordan and Racah coefficients for the quantum algebras $U_q(su(2))$

Let $J_{\pm}$, $q^{\pm J_0}$ be generators of the quantum algebra $U_q(su(2))$. These generators satisfy the standard relations:

$$\begin{aligned} q^{J_0}J_{\pm} &= q^{\pm 1}J_{\pm}q^{J_0}, \qquad [J_+,J_-]=\tfrac{q^{2J_0}-q^{-2J_0}}{q-q^{-1}}\equiv [2J_0]\ , \\ J_{\pm}^{*} &= J_{\mp}\ , \qquad\qquad J_0^{*}=J_0\ , \qquad\qquad q^{*}=q\ (\text{or}\ q^{-1})\ . \end{aligned} \tag{9.1}$$

Here and in what follows we use the notation: $[x]=(q^x-q^{-x})/(q-q^{-1})$. The Hopf structure of $U_q(su(2))$ is given by the following formulas for the comultiplication Δ_q, and the antipode S_q:

$$\begin{aligned} \Delta_q(J_0) &= J_0\otimes 1+1\otimes J_0\ , & S_q(J_0) &= -J_0\ , \\ \Delta_q(J_{\pm}) &= J_{\pm}\otimes q^{J_0}+q^{-J_0}\otimes J_{\pm}\ , & S_q(J_{\pm}) &= -q^{\pm 1}J_{\pm}\ . \end{aligned} \tag{9.2}$$

Let $\{|jm\rangle\}$ be a canonical basis of the $U_q(su(2))$-irreducible representation (IR) with the spin j. These basis vectors satisfy the relations:

$$\begin{aligned} q^{J_0}|jm\rangle &= q^m|jm\rangle\ , \\ J_{\pm}|jm\rangle &= \sqrt{[j\mp m][j\pm m+1]}|jm\pm 1\rangle\ . \end{aligned} \tag{9.3}$$

The vector $|jm\rangle$ can be represented in the following form

$$|jm\rangle = F_{m;j}^{j}|jj\rangle\ , \tag{9.4}$$

where

$$F_{m;j}^{j}=\sqrt{\tfrac{[j+m]!}{[2j]![j-m]!}}\,J_-^{j-m}\ , \tag{9.5}$$

and $|jj\rangle$ is the highest weight vector, i.e.

$$J_+|jj\rangle = 0\ . \tag{9.6}$$

The operator $F_{m;j}^{j}$ is called the lowering operator. We can also introduce the rising operator

$$F_{j;m}^{j}=\sqrt{\tfrac{[j+m]!}{[2j]![j-m]!}}\,J_+^{j-m}\ , \tag{9.7}$$

which has the property

$$|jj\rangle = F^{j}_{j;m}|jm\rangle \ . \tag{9.8}$$

The extremal projector p for $U_q(su(2))$ can be represented in the form

$$p = \sum_{n=0}^{\infty} \frac{(-1)^n \bar{\Gamma}_q(2J_0+2)}{[n]!\bar{\Gamma}_q(2J_0+n+2)}\, J_-^n J_+^n \ , \tag{9.9}$$

where $\bar{\Gamma}_q(x)$ is the modified q-gamma function

$$\bar{\Gamma}_q(x+1) = [x]\Gamma_q(x) \ . \tag{9.10}$$

This function is connected with the standard Heine-Thomae q-gamma function $\Gamma_q(x)$ by the relation $\bar{\Gamma}_q(x) = q^{x(x-1)/4}\Gamma_q(x)$. The extremal projector p satisfies the relations:

$$J_+p = pJ_- = 0 \ , \qquad p^2 = p \ . \tag{9.11}$$

We multiply the extremal projector p by the lowering an rising operators as follows

$$P^{j}_{m;m'} := F^{j}_{m;j}\, p\, F^{j}_{j;m'} \ . \tag{9.12}$$

Below we assume that the operator $P^{j}_{m;m'}$ acts in a vector space of the weight m'. The operator $P^{j}_{m;m'}$ is called the general projection operator.

Let $\{|j_im_i\rangle\}$ be canonical bases of two IRs j_i $(i=1,2)$. Then $\{|j_1m_1\rangle|j_2m_2\rangle\}$ be an 'uncoupled' bases in the representation $j_1\otimes j_2$ of $U_q(su(2))\otimes U_q(su(2))$. In this representation there is another basis $|j_1j_2\!:\!j_3m_3\rangle_q$ which is called a 'coupled' basis with respect to $\Delta_q(U_q(su(2)))$. We can expand the coupled basis in terms of the uncoupled basis $\{|j_1m_1\rangle|j_2m_2\rangle\}$:

$$\big|j_1j_2\!:\!j_3m_3\big\rangle_q = \sum_{m_1,m_2} \big(j_1m_1\, j_2m_2|j_3m_3\big)_q \big|j_1m_1\big\rangle\big|j_2m_2\big\rangle \ , \tag{9.13}$$

where the matrix element $\big(j_1m_1\, j_2m_2|j_3m_3\big)_q$ is called the Clebsch-Gordan coefficient (CGC). After some manipulations we can show that CGC is presented by

$$\big(j_1m_1\, j_2m_2\big|j_3m_3\big)_q = \frac{\langle j_1m_1|\langle j_2m_2|\Delta_q(P^{j_3}_{m_3;j_3})|j_1j_1\rangle|j_2j_3-j_1\rangle}{\sqrt{\langle j_1j_1|\langle j_2\, j_3-j_1|\Delta_q(P^{j_3}_{j_3;j_3})|j_1j_1\rangle|j_2\, j_3-j_1\rangle}} \ . \tag{9.14}$$

This is a formula for calculation of CGCs. Using the explicit expression (9.12) for the general projection operator $P^{j_3}_{m_3;j_3}$, the formulas (9.2) for the comultiplication Δ_q and the actions (9.3) for the generators of $U_q(su(2))$ on the canonical basis vectors $\big|j_im_i\big\rangle$ $(i\!=\!1,2)$ it is not hard to calculate the numerator and the denominator of the right side of (9.14). As result we obtain the following expression for

CGC of the quantum algebra $U_q(su(2))$:

$$\begin{aligned}\big(j_1m_1\, j_2m_2\big|j_3m_3\big)_q &= \delta_{m_1+m_2,m_3}\, q^{-\frac{1}{2}(j_1+j_2-j_3)(j_1+j_2+j_3+1)+j_1m_2-j_2m_1}\\ &\times\sqrt{\frac{[2j_3+1][j_1+j_2-j_3]![j_1-j_2+j_3]![j_1+j_2+j_3+1]![j_2-m_2]![j_3+m_3]!}{[-j_1+j_2+j_3]![j_1+m_1]![j_1-m_1]![j_2+m_2]![j_3-m_3]!}}\\ &\times\sum_n \frac{(-1)^{j_1+j_2-j_3-n}q^{n(j_1+m_1)}[2j_2-n]![j_1+j_2-m_3-n]!}{[n]![j_1+j_2-j_3-n]![j_2-m_2-n]![j_1+j_2+j_3+1-n]!}\ .\end{aligned}\tag{9.15}$$

A total list of different explicit expressions and symmetry properties for the q-CGCs can be found, for example, in [16, 20, 21, 22].

The general formula (9.15) can be expressed in terms of the basic hypergeometric series

$$\begin{aligned}\big(j_1m_1\, j_2m_2\big|j_3m_3\big)_q &= \delta_{m_1+m_2,m_3}\, q^{-\frac{1}{2}(j_1+j_2-j_3)(j_1+j_2+j_3+1)+j_1m_2-j_2m_1}\\ &\times(-1)^{j_1+j_2-j_3}\sqrt{\frac{[2j_3+1][j_1-j_2+j_3]!([2j_2]!)^2}{[j_1+j_2-j_3]![-j_1+j_2+j_3]![j_1+j_2+j_3+1]!}}\\ &\times\sqrt{\frac{[j_3+m_3]!([j_1+j_2-m_3]!)^2}{[j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j_3-m_3]!}}\\ &\times{}_3\Phi_2\left(\begin{matrix}-j_1-j_2+j_3,\ -j_1-j_2-j_3-1,\ -j_2-m_2\\ -2j_2,\quad -j_1-j_2+m_3\end{matrix}\Big|q^2, q^{2(j_1+m_1+1)}\right).\end{aligned}\tag{9.16}$$

We can also obtain the explicit expression of the Racah coefficients or the $6j$-symbols for $U_q(su(2))$ using the extremal projector. Let $\{|j_im_i\rangle\}$ be canonical bases of three IRs j_i $(i=1,2,3)$. The Racah coefficients for $U_q(su(2))$ (or the q-Racah coefficients) are matrix elements of the transformation between two couplings of these representations: $j_1\otimes(j_2\otimes j_3)$ and $(j_1\otimes j_2)\otimes j_3$. i.e.

$$\big|j_1, j_2j_3(j_{23}):jm\big\rangle_q = \sum_{j_{12}} U(j_1j_2jj_3;j_{12}j_{23})_q\,\big|j_1j_2(j_{12}),j_3):jm\big\rangle_q\ ,\tag{9.17}$$

where, for example, the vector $\big|j_1, j_2j_3(j_{23}):jm\big\rangle_q$ corresponds to the first coupling scheme $j_1\otimes(j_2\otimes j_3)$ and it has the form:

$$\begin{aligned}\big|j_1, j_2j_3(j_{23}):jm\big\rangle_q &= \sum_{\substack{m_2m_3\\ m_1m_{23}}}\big(j_1m_1\, j_{23}m_{23}\big|jm\big)_q\big(j_2m_2\, j_3m_3\big|j_{23}m_{23}\big)_q\\ &\qquad\times\big|j_1m_1\big\rangle\big|j_2m_2\big\rangle\big|j_3m_3\big\rangle\ .\end{aligned}\tag{9.18}$$

The q-Racah coefficient $U(j_1j_2jj_3;j_{12}j_{23})_q$ is connected with the q-$6j$-symbol $\{\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\}_q$ by the standard relation

$$U(j_1j_2jj_3;j_{12}j_{23})_q = (-1)^{j_1+j_2+j_3+j}\sqrt{[2j_{12}+1]![2j_{23}+1]!}\left\{\begin{matrix}j_1\ j_2\ j_{12}\\ j_3\ j\ j_{23}\end{matrix}\right\}_q.\tag{9.19}$$

It is not hard to obtain the formula for calculation of the q-$6j$-symbols in terms of projection operators:

$$\begin{aligned}\left\{\begin{matrix}j_1\ j_2\ j_{12}\\ j_3\ \ j\ \ j_{23}\end{matrix}\right\}_q &= \frac{(-1)^{j_1+j_2+j_3+j}}{\sqrt{[2j_{12}+1]![2j_{23}+1]!}}\\ \times&\frac{\langle j_1j{-}j_{23}|\langle j_2j_{23}{-}j_2|\langle j_2j_2|P^{j_{23}}(23)P^{j}(123)P^{j_{12}}(12)|j_1j_{12}{-}j_2\rangle|j_2j_2\rangle|j_3j{-}j_{12}\rangle}{(j_1j_{12}{-}j_2,j_2j_2|_{12}j_{12})_q(j_{12}j_{12},j_3j{-}j_{12}|jj)_q(j_1j{-}j_{23},j_{23}j_{23}|jj)_q(j_2j_2,j_{23}{-}j_2|j_{23}j_{23})_q},\end{aligned}\tag{9.20}$$

where the notations are used: $P^j := P^j_{jj}$, $P^{j_{23}}(23) := \mathrm{id}\otimes\Delta_q(P^{j_{23}})$, $P^j(123) := (\Delta_q\otimes\mathrm{id})\Delta_q(P^j)$, $P^{j_{12}}(12) := \Delta_q(P^{j_{12}})\otimes\mathrm{id}$.

We substitute the explicit expressions for all special CGCs in the denominator of the right side of (9.20) and then we use the actions of the generators of $U_q(su(2))$ on the canonical basis vectors $|j_i m_i\rangle$ $(i=1,2,3)$, and as result we obtain the explicit expression for the q-$6j$-symbol [21, 22]:

$$\begin{aligned}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q &= (-1)^{j_1+j_{12}+j_{23}+j_3}\frac{\nabla(j_1j_2j_{12})_q\nabla(j_2j_3j_{23})_q}{[j_1-j_2+j_{12}]![-j_1+j_2+j_{12}]!}\\ &\times\frac{\nabla(j_{12}j_3j)_q\nabla(j_1j_{23}j)_q[j_{12}+j_3+j+1]![j_1+j_{23}+j+1]!}{[j_2-j_3+j_{23}]![-j_2+j_3+j_{23}]![j_1-j_{23}+j]![-j_{12}+j_3+j]!}\\ &\times\sum_z\frac{(-1)^z[j_1+j-j_{23}+z]![j_3+j-j_{12}+z]![j_{12}+j_{23}+j_2-j-z]!}{[z]![j_1+j_{23}-j-z]![j_{12}+j_3-j-z]![j_2+j-j_{12}-j_{23}+z]![2j+1+z]!}\,,\end{aligned}\tag{9.21}$$

where we use the notation:

$$\nabla(j_1j_2j_3)_q = \sqrt{\frac{[j_1+j_2-j_3]![j_1-j_2+j_3]![-j_1+j_2+j_3]!}{[j_1+j_2+j_3+1]!}}.\tag{9.22}$$

A total list of different explicit expressions and symmetry properties for the q-$6j$-symbols (or the q-Racah coefficients) can be found in the papers [4, 16, 21, 22].

The general formula (9.21) of the q-$6j$-symbol can be expressed in terms of the following basic hypergeometric series

$$\begin{aligned}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q &= (-1)^{j_1+j_{12}+j_{23}+j_3}\frac{\nabla(j_1j_2j_{12})_q\nabla(j_2j_3j_{23})_q\nabla(j_{12}j_3j)_q}{[j_1-j_2+j_{12}]![-j_1+j_2+j_{12}]![j_2-j_3+j_{23}]!}\\ &\times\frac{\nabla(j_1j_{23}j)_q[j_{12}+j_3+j+1]![j_1+j_{23}+j+1]![j_2+j_{12}+j_{23}-j]!}{[-j_2+j_3+j_{23}]![j_1+j_{23}-j]![j_{12}+j_3-j]![j_2+j-j_{12}-j_{23}]![2j+1]!}\\ &\times{}_4\Phi_3\left(\left.{-j_1-j_{23}+j,\,-j_{12}-j_3+j,\,j_1-j_{23}+j+1,\,j_3+j-j_{12}+1 \atop -j_2-j_{12}-j_{23}+j,\,j_2-j_{12}-j_{23}+j+1,\,2j+2}\right| q^2,q^2\right).\end{aligned}\tag{9.23}$$

If we set $j_{12}=j_1+j_2$ in (9.21) we obtain simple explicit expression for the special (so called 'stretched') q-$6j$-symbol:

$$\begin{aligned}\begin{Bmatrix} j_1 & j_2 & j_1+j_2 \\ j_3 & j & j_{23}\end{Bmatrix}_q &= (-1)^{j_1+j_2+j_3+j}\left[\frac{[2j_1]![2j_2]![j_1+j_2+j_3+j+1]!}{[2j_1+2j_2+1]![j_1+j+j_{23}+1]![j_2+j_3+j_{23}+1]!}\right]^{\frac12}\\ &\times\left[\frac{[j_1+j_2-j_3+j]![j_1+j_2+j_3-j]![-j_1+j+j_{23}]![-j_2+j_3+j_{23}]!}{[-j_1-j_2+j_3+j]![j_1+j-j_{23}]![j_1-j+j_{23}]![j_2+j_3-j_{23}]![j_2-j_3+j_{23}]!}\right]^{\frac12}.\end{aligned}\tag{9.24}$$

In the next section we consider a more complicated example of application of the projection operator method for calculation of a general expression for CGCs of the quantum algebra $U_q(su(3))$.

10 Clebsch-Gordan coefficients for the quantum algebra $U_q(su(3))$

Let $\Pi := \{\alpha_1,\alpha_2\}$ be a system of simple roots of the Lie algebra $sl(3)$ ($sl(3) := sl(3,\mathbb{C}) \simeq A_2$), endowed with the following scalar product: $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) =$

2, $(\alpha_1,\alpha_2)=(\alpha_2,\alpha_1)=-1$. The root system Δ_+ of $sl(3)$ consists of the roots $\alpha_1, \alpha_1{+}\alpha_2, \alpha_2$.

The quantum Hopf algebra $U_q(sl(3))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}$, $e_{\pm\alpha_i}$ $(i=1,2)$ with the relations (11.1), (11.2) where $i,j{=}1,2$.

For construction of the composite root vectors $e_{\pm(\alpha_1+\alpha_2)}$ we fix the following normal ordering in Δ_+:

$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_2. \tag{10.1}$$

According to this ordering we set

$$e_{\alpha_1+\alpha_2}:=[e_{\alpha_1},e_{\alpha_2}]_{q^{-1}}\ , \qquad e_{-\alpha_1-\alpha_2}:=[e_{-\alpha_2},e_{-\alpha_1}]_{q}\ . \tag{10.2}$$

Let us introduce another standard notations for the Cartan-Weyl generators:

$$\begin{array}{rclrclrcl} e_{12} &:=& e_{\alpha_1}, & e_{21} &:=& e_{-\alpha_1}, & e_{11}-e_{22} &:=& h_{\alpha_1}, \\ e_{23} &:=& e_{\alpha_2}, & e_{32} &:=& e_{-\alpha_2}, & e_{22}-e_{33} &:=& h_{\alpha_2}, \\ e_{13} &:=& e_{\alpha_1+\alpha_2}, & e_{31} &:=& e_{-\alpha_1-\alpha_2}, & e_{11}-e_{33} &:=& h_{\alpha_1}+h_{\alpha_2}. \end{array} \tag{10.3}$$

The explicit formula for the extremal projector (7.2) specialized to the case of $U_q(sl(3))$ has the form

$$p=p_{12}p_{13}p_{23}\ , \tag{10.4}$$

where the elements p_{ij} $(1\leq i<j\leq 3)$ are given by

$$\begin{array}{rcl} p_{ij} &=& \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{[n]!}\varphi_{ij,n}e_{ij}^{n}e_{ji}^{n}\ , \\ \varphi_{ij,n} &=& q^{-(j-i-1)n}\Big\{\prod\limits_{s=1}^{n}[e_{ii}-e_{jj}+j-i+s]\Big\}^{-1}\ . \end{array} \tag{10.5}$$

The extremal projector p satisfies the relations:

$$e_{ij}p=pe_{ji}=0\quad(i<j)\ , \qquad p^2=p\ . \tag{10.6}$$

The quantum algebra $U_q(su(3))$ can be considered as the quantum algebra $U_q(sl(3))$ endowed with the additional Cartan involution *:

$$h_{\alpha_i}^{*}=h_{\alpha_i}\ , \qquad e_{\pm\alpha_i}^{*}=e_{\mp\alpha_i}\ , \qquad q^*=q\ (\text{or}\ q^{-1})\ . \tag{10.7}$$

Let $(\lambda\mu)$ be a finite-dimensional IR of $U_q(su(3))$ with the highest weight $(\lambda\mu)$ (λ and μ are nonnegative integers). The vector of the highest weight, denoted by the symbol $\big|(\lambda\mu)h\big\rangle$, satisfy the relations

$$\begin{array}{rcl} h_{\alpha_1}\big|(\lambda\mu)h\big\rangle &=& \lambda\big|(\lambda\mu)h\big\rangle\ , \qquad h_{\alpha_2}\big|(\lambda\mu)h\big\rangle=\mu\big|(\lambda\mu)h\big\rangle\ , \\ e_{ij}\big|(\lambda\mu)h\big\rangle &=& 0 \qquad (i<j)\ . \end{array} \tag{10.8}$$

Labeling of another basis vectors in the IR $(\lambda\mu)$ depends on the choice of subalgebras of $U_q(u(3))$ (in other words which reduction chain from $U_q(u(3))$ to subalgebras is chosen). Here we use the Gelfand-Tsetlin reduction chain:

$$U_q(su(3))\supset U_q(u_Y(1))\otimes U_q(su_T(2))\supset U_q(u_{T_0}(1))\ , \tag{10.9}$$

where the subalgebra $U_q(su_T(2))$ is generated by the elements

$$T_+ := e_{23} , \qquad T_- := e_{32} , \qquad T_0 := \frac{1}{2}(e_{22} - e_{33}) , \tag{10.10}$$

the subalgebra $U_q(u_{T_0}(1))$ is generated by q^{T_0}, and the subalgebra $U_q(u_Y(1))$ is generated by q^Y where[5]:

$$Y = -\frac{1}{3}\bigl(2h_{\alpha_1} + h_{\alpha_2}\bigr) . \tag{10.11}$$

In the case of the reduction chain (10.9) the basis vectors of IR $(\lambda\mu)$ are denoted by

$$\bigl|(\lambda\mu)jtt_z\bigr\rangle . \tag{10.12}$$

Here the quantum number set jtt_z characterize the hypercharge y and the T-spin t and its projection t_z:

$$\begin{array}{rcl} q^{T_0}\bigl|(\lambda\mu)jtt_z\bigr\rangle &=& q^{t_z}\bigl|(\lambda\mu)jtt_z\bigr\rangle , \\ T_\pm\bigl|(\lambda\mu)jtt_z\bigr\rangle &=& \sqrt{[t\mp t_z][t\pm t_z+1]}\bigl|(\lambda\mu)jtt_z\pm1\bigr\rangle , \\ q^Y\bigl|(\lambda\mu)jtt_z\bigr\rangle &=& q^y\bigl|(\lambda\mu)jtt_z\bigr\rangle , \end{array} \tag{10.13}$$

where the parameter j is connected with the eigenvalue y of the "hypercharge" operator Y as follows

$$y = -\frac{1}{3}\bigl(2\lambda + \mu\bigr) + 2j . \tag{10.14}$$

We can show (see [18, 19]) that the quantum numbers jt are taken all nonnegative integers and half-integers such that the sum $\frac{1}{2}\mu + j + t$ is an integer and they are subjected to the constraints:

$$\left\{ \begin{array}{rcl} \frac{1}{2}\mu + j - t &\geq& 0 , \\ \frac{1}{2}\mu - j + t &\geq& 0 , \\ -\frac{1}{2}\mu + j + t &\geq& 0 , \\ \frac{1}{2}\mu + j + t &\geq& \lambda + \mu . \end{array} \right. \tag{10.15}$$

For every fixed t the projection t_z runs the values $t_z = -t, -t+1, \ldots, t-1, t$. It is not hard to show that the orthonormalized vectors (10.12) can be represented in the following form

$$\bigl\{\bigl|(\lambda\mu)jtt_z\bigr\rangle := N_{jt}^{(\lambda\mu)} P^{\,t}_{t_z;t}\,\mathcal{R}^{\,j}_{\frac{1}{2}\mu-t}\bigl|(\lambda\mu)h\bigr\rangle\bigr\} , \tag{10.16}$$

where $P^{\,t}_{t_z;t'_z}$ is the general projection operator of the type (9.12) for the quantum algebra $U_q(su_T(2))$, the element $\mathcal{R}^{\,j}_{\frac{1}{2}\mu-t}$ is a component of the irreducible tensor operator of rank j, $(-j \leq j_z \leq j)$:

$$\mathcal{R}^{\,j}_{j_z} = \sqrt{\frac{[2j]!}{[j-j_z]![j+j_z]!}}(-1)^{j_z} q^{2j^2-j+j_z}\, e_{21}^{j-j_z} e_{31}^{j+j_z} q^{-jh_{\alpha_1}-(j+j_z)T_0} , \tag{10.17}$$

[5]In the classical non-deformed case in the elementary particle theory the subalgebra $su_T(2)$ is called the T-spin algebra and the element Y is the hypercharge operator.

the normalizing factor $N_{jt}^{(\lambda\mu)}$ has the form

$$
\begin{aligned}
N_{jt}^{(\lambda\mu)} &= (-1)^{t-\frac{1}{2}\mu}\, q^{(j+\frac{1}{2}\mu-t)(j+t)-2j^2+j\lambda-\mu+2t} \\
&\times \sqrt{\frac{[\lambda+\frac{1}{2}\mu-j+t+1]![\lambda+\frac{1}{2}\mu-j-t]![\frac{1}{2}\mu+j+t+1]![\frac{1}{2}\mu-j+t]!}{[\lambda]![\mu]![\lambda+\mu+1]![2j]![2t+1]!}}\ .
\end{aligned}
\tag{10.18}
$$

The operator

$$
F_{jtt_z;h}^{(\lambda\mu)} = N_{jt}^{(\lambda\mu)} P_{t_z;t}^{\,t} \mathcal{R}_{\frac{1}{2}\mu-t}^{\,j} \tag{10.19}
$$

is called the lowering operator and the conjugate operator

$$
F_{h;jtt_z}^{(\lambda\mu)} = (F_{jtt_z;h}^{(\lambda\mu)})^{*} \tag{10.20}
$$

is called the rising operator of $U_q(su(3))$.

If the extremal operator p acts on a vector of the weight $(\lambda\mu)$ with respect to the operators h_{α_1} and h_{α_2} then we write $p^{(\lambda\mu)}$. We multiply the extremal projector $p^{(\lambda\mu)}$ by the lowering and rising operators as follows

$$
P_{jtt_z;j't't'_z}^{(\lambda\mu)} = F_{jtt_z;h}^{(\lambda\mu)}\, p^{(\lambda\mu)} F_{h;j't't'_z}^{(\lambda\mu)}\ , \tag{10.21}
$$

and we assume that the resulting operator acts on a vector space of the fixed weight $(-\frac{1}{3}(2\lambda+\mu)+2j, t_z)$ with respect to the operators Y and T_0. The operator $P_{jtt_z;j't't'_z}^{(\lambda\mu)}$ is called the general projection operator of $U_q(su(3))$.

For convenience we introduce the short notations: $\Lambda := (\lambda\mu)$ and $\gamma := jtt_z$ and therefore the basis vector (10.12) will be denoted by $\big|\Lambda\gamma\big\rangle$. Let $\{|\Lambda_i\gamma_i\big\rangle\}$ be Gelfand-Tsetlin bases of two IRs Λ_i $(i = 1, 2)$. Then $\{|\Lambda_1\gamma_1\big\rangle|\Lambda_2\gamma_2\big\rangle\}$ be a uncoupled bases in the representation $\Lambda_1\otimes\Lambda_2$ of $U_q(su(3))\otimes U_q(su(3))$. In this representation there is another coupled basis $|\Lambda_1\Lambda_2\colon s\Lambda_3\gamma_3\big\rangle_q$ with respect to $\Delta_q(U_q(su(3)))$ where the index s classifies multiple representations Λ. We can expand the coupled basis in terms of the uncoupled basis $\{|\Lambda_1\gamma_1\big\rangle|\Lambda_2\gamma_2\big\rangle\}$:

$$
\big|\Lambda_1\Lambda_2\colon s\Lambda_3\gamma_3\big\rangle_q = \sum_{\gamma_1,\gamma_2} \big(\Lambda_1\gamma_1\,\Lambda_2\gamma_2|s\Lambda_3\gamma_3\big)_q \big|\Lambda_1\gamma_1\big\rangle\big|\Lambda_2\gamma_2\big\rangle\ , \tag{10.22}
$$

where the matrix element $\big(\Lambda_1\gamma_1\,\Lambda_2\gamma_2|\Lambda_3\gamma_3\big)_q$ is the Clebsch-Gordan coefficient of $U_q(su(3))$. In just the same way as for the non-quantized Lie algebra $su(3)$ (see [18, 19]) we can show that any CGC of $U_q(su(3))$ can be represented in terms of the linear combination of the matrix elements of the projection operator (10.21)

$$
\big(\Lambda_1\gamma_1\,\Lambda_2\gamma_2|s\Lambda_3\gamma_3\big)_q = \sum_{\gamma'_2} C(\gamma'_2)\big\langle\Lambda_1\gamma_1\big|\big\langle\Lambda_2\gamma_2\big|\Delta_q(P_{\gamma_3;h}^{\Lambda_3})\big|\Lambda_1 h > \big|\Lambda_2\gamma'_2\big\rangle\ . \tag{10.23}
$$

Classification of multiple representations Λ_3 in the representation $\Lambda_1\otimes\Lambda_2$ is special problem and we shall not discuss it here. For the non-deformed algebra $su(3)$ this problem was considered in details in [18, 19]. Concerning the matrix elements in

the right-side of (10.23) we give here an explicit expression for the more general matrix element:

$$\big\langle\Lambda_1\gamma_1\big|\big\langle\Lambda_2\gamma_2\big|\Delta_q(P^{\Lambda_3}_{\gamma_3;\gamma_3'})\big|\Lambda_1\gamma_1'>\big|\Lambda_2\gamma_2'\big\rangle\;. \tag{10.24}$$

Using a tensor form of the projection operator (10.21) and the Wigner-Racah calculus for the subalgebra $U_q(su(2))$ [20] it is not hard to obtain the following result (see [5]):

$$\begin{aligned}
&\big\langle\Lambda_1\gamma_1\big|\big\langle\Lambda_2\gamma_2\big|\Delta_q(P^{\Lambda}_{\gamma_3;\gamma_3'})\big|\Lambda_1\gamma_1'\big\rangle\big|\Lambda_2\gamma_2'\big\rangle\\
&=\big(t_1t_{1z}\,t_2t_{2z}\big|t_3t_{3z}\big)_q\big(t_1t_{1z}'\,t_2t_{2z}'\big|t_3't_{3z}'\big)_q\;\;A\sum_{j_1''j_2''t_1''t_2''t_3''}C_{j_1''j_2''t_1''t_2''t_3''}\\
&\times\left\{\begin{matrix} j_1-j_1'' & j_2-j_2'' & j_1+j_2-j_1''-j_2''\\ t_1'' & t_2'' & t_3''\\ t_1 & t_2 & t_3\end{matrix}\right\}_q\left\{\begin{matrix} j_1'-j_1'' & j_2'-j_2'' & j_1'+j_2'-j_1''-j_2''\\ t_1'' & t_2'' & t_3''\\ t_1' & t_2' & t_3'\end{matrix}\right\}_q\;.
\end{aligned}\tag{10.25}$$

Here

$$\begin{aligned}
A&=[\lambda+1][\mu+1][\lambda+\mu+2]\\
&\quad\times\left[\frac{[2t_1+1][2t_2+1][2j_1+1]![2j_2+1]![\lambda_3+\frac{1}{2}\mu_3-j_3+t_3+1]![\lambda_3+\frac{1}{2}\mu_3-j_3-t_3]!}{[\lambda_1+\frac{1}{2}\mu_1-j_1+t_1+1]![\lambda_1+\frac{1}{2}\mu_1-j_1-t_1]![\lambda_2+\frac{1}{2}\mu_2-j_2+t_2+1]![\lambda_2+\frac{1}{2}\mu_2-j_2-t_2]![2j_3]!}\right]^{\frac{1}{2}}\\
&\quad\times\left[\frac{[2t_1'+1][2t_2'+1][2j_1'+1]![2j_2'+1]![\lambda_3+\frac{1}{2}\mu_3-j_3'+t_3'+1]![\lambda_3+\frac{1}{2}\mu_3-j_3'-t_3']!}{[\lambda_1+\frac{1}{2}\mu_1-j_1'+t_1'+1]![\lambda_1+\frac{1}{2}\mu_1-j_1'-t_1']![\lambda_2+\frac{1}{2}\mu_2-j_2'+t_2'+1]![\lambda_2+\frac{1}{2}\mu_2-j_2'-t_2']![2j_3']!}\right]^{\frac{1}{2}},
\end{aligned}\tag{10.26}$$

the coefficient $C_{j_1''j_2''t_1''t_2''t_3''}$ does not contain any 'inner' summation (without summation!) and has the form

$$\begin{aligned}
&C_{j_1''j_2''t_1''t_2''t_3''}=\frac{(-1)^{2(j_1+j_2+j_3'-j_1''-j_2'')}q^{\psi}[2(j_1+j_2-j_1''-j_2'')+1]![2(j_1'+j_2'-j_1''-j_2'')+1]![2t_3''+1]}{[2j_1'']![2j_2'']![2j_1-2j_1'']![2j_2-2j_2'']![2j_1'-2j_2'']![2j_2'-2j_2'']![2(j_1+j_2-j_3-j_1''-j_2'')]!}\\
&\times\frac{[\lambda_1+\frac{1}{2}\mu_1-j_1''+t_1''+1]![\lambda_1+\frac{1}{2}\mu_1-j_1''-t_1'']![\lambda_2+\frac{1}{2}\mu_2-j_2''+t_2''+1]![\lambda_2+\frac{1}{2}\mu_2-j_2''-t_2'']![2t_1''+1][2t_2''+1]}{[\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_1''-j_2''+t_3''+2]![\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3'-j_1''-j_2''-t_3''+1]!}\\
&\times\left\{\begin{matrix} j_1-j_1'' & j_1'' & j_1\\ \frac{1}{2}\mu_1 & t_1 & t_1''\end{matrix}\right\}_q\left\{\begin{matrix} j_2-j_2'' & j_2'' & j_2\\ \frac{1}{2}\mu_2 & t_2 & t_2''\end{matrix}\right\}_q\left\{\begin{matrix} j_3 & j_1+j_2-j_3-j_1''-j_2'' & j_1+j_2-j_1''-j_2''\\ t_3'' & t_3 & \frac{1}{2}\mu_3\end{matrix}\right\}_q\\
&\times\left\{\begin{matrix} j_1'-j_1'' & j_1'' & j_1'\\ \frac{1}{2}\mu_1 & t_1' & t_1''\end{matrix}\right\}_q\left\{\begin{matrix} j_2'-j_2'' & j_2'' & j_2'\\ \frac{1}{2}\mu_2 & t_2' & t_2''\end{matrix}\right\}_q\left\{\begin{matrix} j_3' & j_1'+j_2'-j_3'-j_1''-j_2'' & j_1'+j_2'-j_1''-j_2''\\ t_3'' & t_3' & \frac{1}{2}\mu_3\end{matrix}\right\}_q,
\end{aligned}\tag{10.27}$$

where

$$\begin{aligned}
\psi&=\sum_{i=1}^{2}\Big(2\varphi(\lambda_i,\mu_i,j_i'',t_i'')-\varphi(\lambda_i,\mu_i,j_i,t_i)-\varphi(\lambda_i,\mu_i,j_i',t_i')\\
&\quad-t_i(t_i+1)-t_i'(t_i'+1)\Big)-2\varphi(\lambda_3,\mu_3,j_3'',t_3'')+\varphi(\lambda_3,\mu_3,j_3,t_3)\\
&\quad+\varphi(\lambda_3,\mu_3,j_3',t_3')+j_3''(4\lambda_3+2\mu_3+2)-2t_3''(t_3''-1)-2\mu_3\\
&\quad-(j_2+j_2'-2j_2'')(2\lambda_1+\mu_1-6j_1'')-(j_3+j_3'')(j_3+j_3''+1)\\
&\quad-(j_3'+j_3'')(j_3'+j_3''+1)+4(j_1-j_1'')(j_2-j_2'')+4(j_1'-j_1'')(j_2'-j_2'')\;,
\end{aligned}\tag{10.28}$$

$$\begin{aligned}\varphi(\lambda,\mu,j,t) &= \tfrac{1}{2}(\tfrac{1}{2}\mu+j-t)(\tfrac{1}{2}\mu+j+t-3)+j(\lambda-2j+1)\ ,\\ j_3'' &= j_1+j_2-j_3-j_1''-j_2''=j_1'+j_2'-j_3'-j_1''-j_2''.\end{aligned}\tag{10.29}$$

A detailed proof of the formulas (10.25)-(10.29) is given in [5].

The q-9j-symbol of $U_q(su(2))$ in (10.25) can be expressed in terms of q-6j-symbols. This expression has the form:

$$\begin{aligned}\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j_4 & j_{34}\\ j_{13} & j_{24} & j\end{matrix}\right\}_q &= \sum_z (-1)^{2z} q^{c(z)+c(j_{24})+c(j_{34})+c(j)}[2z+1]\\ &\quad\times\left\{\begin{matrix} j_1\ j_2\ j_{12}\\ z\ j_3\ j_{13}\end{matrix}\right\}_q \left\{\begin{matrix} j_3\ j_4\ j_{34}\\ j\ j_{12}\ z\end{matrix}\right\}_q \left\{\begin{matrix} j_{13}\ j_{24}\ j\\ j_4\ z\ j_2\end{matrix}\right\}_q ,\end{aligned}\tag{10.30}$$

where $c(j):=j(j+1)$. In our case $j_{12}=j_1+j_2$ and we have in the right side of (10.30) one particular ('stretched') q-6j-symbol (9.24) which does not contain a summation. Therefore each q-9j-symbol in the right-side of (10.25) is expressed in terms of a linear combination of products of the general q-6j-symbols. Since the general q-6j-symbol can be expressed in terms of the basic hypergeometric series ${}_4\Phi_3\left({\cdots \atop \cdots};q,q\right)$ therefore the q-9j-symbol is expressed in terms of a linear combination of products of two basic hypergeometric series ${}_4\Phi_3\left({\cdots \atop \cdots};q,q\right)$.

Thus, *the general matrix element (10.25) can be expressed in terms of a linear combination of products of four basic hypergeometric series* ${}_4\Phi_3\left({\cdots \atop \cdots};q,q\right)$.

11 Gelfand-Tsetlin basis for $U_q(u(n))$

Let $\Pi:=\{\alpha_1,\ldots,\alpha_{n-1}\}$ be a system of simple roots of the Lie algebra $sl(n)$ ($sl(n):=sl(n,\mathbb{C})\simeq A_{n-1}$) endowed with the following scalar product: $(\alpha_i,\alpha_j)=(\alpha_j,\alpha_i)$, $(\alpha_i,\alpha_i)=2$, $(\alpha_i,\alpha_{i+1})=-1$, $(\alpha_i,\alpha_j)=0$ (($|i-j|>1$).

The quantum Hopf algebra $U_q(sl(n))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}$, $e_{\pm\alpha_i}$ ($i=1,2,\ldots,n-1$) with the defining relations:

$$\begin{aligned} q^{h_{\alpha_i}}q^{-h_{\alpha_i}} &= q^{-h_{\alpha_i}}q^{h_{\alpha_i}}=1\ , \qquad q^{h_{\alpha_i}}q^{h_{\alpha_j}}=q^{h_{\alpha_j}}q^{h_{\alpha_i}}\ ,\\ q^{h_{\alpha_i}}e_{\pm\alpha_j}q^{-h_{\alpha_i}} &= q^{\pm(\alpha_i,\alpha_j)}e_{\pm\alpha_j}\ , \qquad [e_{\alpha_i},e_{-\alpha_j}]=\delta_{ij}\,[h_{\alpha_i}]\ ,\\ [e_{\pm\alpha_i},e_{\pm\alpha_j}] &= 0 \qquad (|i-j|\geq 2)\ ,\\ [[e_{\pm\alpha_i}e_{\pm\alpha_j}]_q e_{\pm\alpha_j}]_q &= 0 \qquad (|i-j|=1)\ ,\end{aligned}\tag{11.1}$$

$$\begin{aligned}\Delta_q(h_{\alpha_i}) &= h_{\alpha_i}\otimes 1+1\otimes h_{\alpha_i}, & S_q(h_{\alpha_i}) &= -h_{\alpha_i},\\ \Delta_q(e_{\pm\alpha_i}) &= e_{\pm\alpha_i}\otimes q^{\frac{h_{\alpha_i}}{2}}+q^{-\frac{h_{\alpha_i}}{2}}\otimes e_{\pm\alpha_i}, & S_q(e_{\alpha_i}) &= -q^{\pm 1}e_{\alpha_i}.\end{aligned}\tag{11.2}$$

Below we shall use another basis in the Cartan subalgebra of the algebra $sl(n)$

$(U_q(sl(n))$. Namely we set

$$
\begin{array}{rcl}
e_{11} & = & \frac{1}{n}\big((n-1)h_{\alpha_1}+(n-2)h_{\alpha_2}+\cdots+2h_{\alpha_{n-2}}+h_{\alpha_{n-1}}+N\big)\ , \\
e_{22} & = & \frac{1}{n}\big((n-1)h_{\alpha_1}+(n-2)h_{\alpha_2}+\cdots+2h_{\alpha_{n-2}}+h_{\alpha_{n-1}}+N\big)-h_{\alpha_1}\ , \\
\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
e_{ii} & = & \frac{1}{n}\big((n-1)h_{\alpha_1}+(n-2)h_{\alpha_2}+\cdots+2h_{\alpha_{n-2}}+h_{\alpha_{n-1}}+N\big)-\sum\limits_{k=1}^{i-1}h_{\alpha_k}\ , \\
\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
e_{nn} & = & \frac{1}{n}\big(-h_{\alpha_1}-2h_{\alpha_2}-\cdots-(n-2)h_{\alpha_{n-2}}-(n-1)h_{\alpha_{n-1}}+N\big)\ .
\end{array}
\tag{11.3}
$$

Here N is a central element of g $(U_q(g))$, which is equal to 0 for the case $g=sl(n)$ and $N\neq 0$ for $g=gl(n)$. It is easy to see that

$$
\begin{array}{rcl}
h_{\alpha_i} & = & e_{ii}-e_{i+1i+1} \qquad (i=1,\ldots,n-1)\ , \\
N & = & e_{11}+e_{22}+\ldots+e_{nn}\ .
\end{array}
\tag{11.4}
$$

Dual elements to the ones e_{ii} $(i=1,2,\ldots,n)$ will be denoted by ϵ_i $(i=1,2,\ldots,n)$: $\epsilon_i(e_{jj})=(\epsilon_i,\epsilon_j)=\delta_{ij}$. In the terms of ϵ_i the positive root system Δ_+ of $sl(n)$ is presented as follows

$$
\Delta_+=\{\epsilon_i-\epsilon_j\,|\,1\leq i<j\leq n\}\ , \tag{11.5}
$$

where $\epsilon_i-\epsilon_{i+1}$ are the simple roots: $\alpha_i=\epsilon_i-\epsilon_{i+1}$ $(i=1,2,\ldots,n-1)$. For the root vectors $e_{\epsilon_i-\epsilon_j}$ $(i\neq j)$ the another standard notations are also used

$$
e_{ij}:=e_{\epsilon_i-\epsilon_j}\ , \qquad e_{ji}:=e_{\epsilon_j-\epsilon_i} \qquad (1\leq i<j\leq n)\ . \tag{11.6}
$$

In particular, the elements e_{ii+1}, e_{i+1i} are the Chevalley generators: $e_{ii+1}=e_{\alpha_i}$, $e_{i+1i}=e_{-\alpha_i}$ $(i=1,\ldots,n-1)$.

For construction of the composite root vectors e_{ij} $(j\neq i\pm1)$ we fix the following normal ordering of the positive root system Δ_+ (see [26])

$$
(\epsilon_1-\epsilon_2),(\epsilon_1-\epsilon_3,\epsilon_2-\epsilon_3),\ldots,(\epsilon_1-\epsilon_i,\ldots,\epsilon_{i-1}-\epsilon_i),\ldots,(\epsilon_1-\epsilon_n,\ldots,\epsilon_1-\epsilon_n). \tag{11.7}
$$

According to this ordering we set

$$
e_{ij}:=[e_{ik},e_{kj}]_{q^{-1}}\ , \qquad e_{ji}:=[e_{jk},e_{ki}]_q \qquad (1\leq i<k<j\leq n)\ . \tag{11.8}
$$

It should be stressed that the structure of the composite root vectors (11.8) is independent of the choice of the index k in the r.h.s. of the definition (11.8). In particular, one has

$$
\begin{array}{rclcll}
e_{ij} & := & [e_{ii+1},e_{i+1j}]_{q^{-1}} & = & [e_{ij-1},e_{j-1j}]_{q^{-1}} & (1\leq i<j\leq n)\ , \\
e_{ji} & := & [e_{ji+1},e_{i+1i}]_q & = & [e_{jj-1},e_{j-1i}]_q & (1\leq i<j\leq n)\ .
\end{array}
\tag{11.9}
$$

The explicit formula for the extremal projector (7.2) specialized to the case of $U_q(sl(n))$ has the form

$$
\begin{array}{rcl}
p(U_q(sl(n)) & = & p(U_q(sl(n-1))(p_{1n}p_{2n}\cdots p_{n-2n}p_{n-1n}) \\
 & = & p_{12}(p_{13}p_{23})\cdots(p_{1i}\cdots p_{ii+1})\cdots(p_{1n}\cdots p_{n-1n})\ ,
\end{array}
\tag{11.10}
$$

where the elements p_{ij} are given by the formulas (10.5) with $(1 \le i < j \le n)$. The extremal projector $p := p(U_q(sl(n))$ satisfies the eqs. (10.6) with $(1 \le i < j \le n)$.

The quantum algebra $U_q(su(n))$ can be considered as the quantum algebra $U_q(sl(n))$ endowed with the additional Cartan involution *:

$$e_{ij}^* = e_{ji} \ , \qquad q^* = q \ (\text{or } q^{-1}) \ . \tag{11.11}$$

Since the quantum algebra $U_q(su(n))$ can be interpreted as the algebra $U_q(u(n))$ with the central element $N = 0$ and the inner structure of its representations is more easily described in terms of $U_q(u(n))$, we shall consider the quantum algebra $U_q(u(n))$.

Let V^{λ_n} be a finite-dimensional IR of $U_q(u(n))$ with the highest weight $\lambda_n := (\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn})$ where $\lambda_{in} - \lambda_{i+1n}$ $(i = 1, \ldots, n-1)$ are nonnegative integers. The vector of the highest weight, denoted by the symbol $|\lambda_n\rangle$, satisfy the relations

$$\begin{aligned} q^{h_{ii}}|\lambda_n\rangle &= q^{\lambda_{in}}|\lambda_n\rangle \qquad (1 \le i \le n) \ , \\ e_{ij}|\lambda_n\rangle &= 0 \qquad\quad (i < j) \ . \end{aligned} \tag{11.12}$$

Labeling of another basis vectors in IR V^{λ_n} depends on choice of subalgebras of $U_q(u(n))$ (in other words which reduction chain from $U_q(u(n))$ to subalgebras is chosen). Here we use the 'so called' Gelfand-Tsetlin reduction chain:

$$U_q(u(n)) \supset U_q(u(n-1)) \supset \ldots \supset U_q(u(k)) \supset \ldots U_q(u(1)) \ , \tag{11.13}$$

where the subalgebra $U_q(u(k))$ is generated by e_{ij} with $i, j = 1, 2, \ldots, k$.

The following theorem can be proved.

Theorem 11.1 *In the $U_q(u(n))$-module V^{λ_n} there is the orthogonal Gelfand-Tsetlin basis consisting of all vectors of the form*

$$\begin{aligned} |\lambda\rangle &:= \left| \begin{array}{lllll} \lambda_{1n} & \lambda_{2n} & \ldots & \lambda_{n-1n} & \lambda_{nn} \\ \lambda_{1n-1} & \lambda_{2n-1} & \ldots & \lambda_{n-1n-1} & \\ \ldots & \ldots & \ldots & & \\ \lambda_{12} & \lambda_{22} & & & \\ \lambda_{11} & & & & \end{array} \right\rangle \\ &= F_-(\lambda_1;\lambda_2)F_-(\lambda_1;\lambda_2)\cdots F_-(\lambda_{n-1};\lambda_n)|\lambda_n\rangle \ , \end{aligned} \tag{11.14}$$

where the numbers λ_{ij} satisfy the standard inequalities ('between conditions') for the Lie algebra $u(n)$, i.e.

$$\lambda_{ij+1} \ge \lambda_{ij} \ge \lambda_{i+1j+1} \qquad \text{for } \ 1 \le i \le j \le n-1 \ . \tag{11.15}$$

The lowering operators $F_-(\lambda_k; \lambda_{k+1})$, $(k = 1, 2 \ldots, n-1)$, are given by

$$F_-(\lambda_k;\lambda_{k+1}) = \mathcal{N}(\lambda_k;\lambda_{k+1})\, p(U_q(u(k))) \prod_{i=1}^{k} (e_{k+1i})^{\lambda_{ik+1}-\lambda_{ik}} \ , \tag{11.16}$$

$$
\begin{aligned}
\mathcal{N}(\lambda_k;\lambda_{k+1}) \;=\; & \Big\{ \prod_{i=1}^{k} \tfrac{[l_{ik}-l_{k+1\,k+1}-1]!}{[l_{ik+1}-l_{ik}]![l_{ik+1}-l_{k+1\,k+1}-1]!} \times \\
& \times \prod_{1\le i<j\le k} \tfrac{[l_{ik+1}-l_{jk}]![l_{ik}-l_{jk+1}-1]!}{[l_{ik}-l_{jk}]![l_{ik+1}-l_{jk+1}-1]!} \Big\}^{\frac{1}{2}}
\end{aligned} \tag{11.17}
$$

where $l_{ij}:=\lambda_{ij}-i$.

The explicit form of the basis vectors (11.14) allows to calculate the actions of the Cartan-Weyl generators on these vectors.

For the Cartan elements $q^{e_{ii}}$, $(i=1,2,\ldots,n)$, we easy find that

$$
q^{e_{ii}}\big|\lambda\big\rangle = q^{S_i-S_{i-1}}\big|\lambda\big\rangle \;. \tag{11.18}
$$

Here $S_i=\sum_{j=1}^{i}\lambda_{ji}$, where the numbers $\lambda_{1,i},\lambda_{2i},\ldots,\lambda_{ii}$ are ones in the i-th row of the pattern λ of the vector (11.14), $S_0=0$. The computation of the action of the generators e_{ij} for $i\neq j$ is more difficult. The procedure of this computation goes as follows. First of all, using the explicit expression (11.14) for the basis vector $\big|\lambda\big\rangle$, we determine the transformation of the basis under the action of the generators e_{ii+1} and e_{i+1i} for all $i=1,2,\ldots,n-1$. Then using the inductive definition (11.8) for the generators e_{ij} we find their action for all $i\neq j$. The result reads as follows.

Theorem 11.2 *The Cartan-Weyl generators* e_{kk+s} *and* e_{k+sk} $(s=1,2,\ldots,n-k)$ *of the quantum algebra* $U_q(u(n))$ *act on the Gelfand-Tsetlin basis (11.14) according to relations:*

$$
e_{kk+s}\big|\lambda\big\rangle = \sum_{j_i,j_2,\ldots,j_s}\Big\langle\lambda+\sum_{p=1}^{s}\epsilon_{j_p,k+p-1}\Big|e_{kk+s}\big|\lambda\big\rangle\,\Big|\lambda+\sum_{p=1}^{s}\epsilon_{j_p,k+p-1}\Big\rangle, \tag{11.19}
$$

$$
e_{k+sk}\big|\lambda\big\rangle = \sum_{j_i,j_2,\ldots,j_s}\Big\langle\lambda-\sum_{p=1}^{s}\epsilon_{j_p,k+p-1}\Big|e_{k+sk}\big|\lambda\big\rangle\,\Big|\lambda-\sum_{p=1}^{s}\epsilon_{j_p,k+p-1}\Big\rangle. \tag{11.20}
$$

Here

$$
\begin{aligned}
\Big\langle\lambda+\sum_{p=1}^{s}\epsilon_{j_p,k+p-1}\Big|e_{kk+s}\big|\lambda\big\rangle \;=\; & A_s(\lambda)q^{l_{j_1k}-l_{j_sk+s-1}} \\
& \times\prod_{r=1}^{s}\big\langle\lambda+\epsilon_{j_r,k+r-1}\big|e_{k+r-1k+r}\big|\lambda\big\rangle,
\end{aligned} \tag{11.21}
$$

$$
\begin{aligned}
\Big\langle\lambda-\sum_{p=1}^{s}\epsilon_{j_p,k+p-1}\Big|e_{k+sk}\big|\lambda\big\rangle \;=\; & A_{s-1}(\lambda)q^{l_{j_sk+s-1}-l_{j_1k}} \\
& \times\prod_{r=1}^{s}\big\langle\lambda-\epsilon_{j_r,k+r-1}\big|e_{k+rk+r-1}\big|\lambda\big\rangle,
\end{aligned} \tag{11.22}
$$

where

$$\left\langle\lambda+\epsilon_{j_r,k+r-1}\middle|e_{k+r-1\,k+r}\middle|\lambda\right\rangle = \left\{-\frac{\prod\limits_{i=1}^{k+r}[l_{i\,k+r}-l_{j_r\,k+r-1}]\prod\limits_{i=1}^{k+r-2}[l_{i\,k+r-2}-l_{j_r\,k+r-1}-1]}{\prod\limits_{\substack{i=1\\i\neq j_r}}^{k+r-1}[l_{i\,k+r-1}-l_{j_r\,k+r-1}][l_{i\,k+r-1}-l_{j_r\,k+r-1}-1]}\right\}^{\frac{1}{2}}, \tag{11.23}$$

$$\left\langle\lambda+\epsilon_{j_r,k+r-1}\middle|e_{k+r-1\,k+r}\middle|\lambda\right\rangle = \left\{-\frac{\prod\limits_{i=1}^{k+r}[l_{i\,k+r}-l_{j_r\,k+r-1}+1]\prod\limits_{i=1}^{k+r-2}[l_{i\,k+r-2}-l_{j_r\,k+r-1}]}{\prod\limits_{\substack{i=1\\i\neq j_r}}^{k+r-1}[l_{i\,k+r-1}-l_{j_r\,k+r-1}+1][l_{i\,k+r-1}-l_{j_r\,k+r-1}]}\right\}^{\frac{1}{2}}, \tag{11.24}$$

$$A_s(\lambda) = \prod_{r=1}^{s}\frac{\mathrm{sign}(l_{j_{r+1}\,k+r}-l_{j_r\,k+r-1})}{\{[l_{j_{r+1}\,k+r}-l_{j_r\,k+r-1}][l_{j_{r+1}\,k+r}-l_{j_r\,k+r-1}-1]\}^{\frac{1}{2}}}\,, \tag{11.25}$$

$\mathrm{sign}(x)=1$ *for* $x\geq 0$ *and* $\mathrm{sign}(x)=-1$ *for* $x<0$.

In (11.19), (11.20) each summation index j_r runs over integers $1,2,\ldots,k+r-1$. The symbol $\left|\epsilon_{ij}\right\rangle$ means the Gelfand-Tsetlin patter, which has zeros everywhere, except 1 on the place (ij). The sum of the Gelfand-Tsetlin patterns is given with the sums of the corresponding labels, as a sum of matrices.

In the case $s=1$ the formulas (11.19)-(11.25) coincide with the results of [8], where they have been given for the first time, however without a proof.

At the limit $q\rightarrow 1$ the formulas (11.14)–(11.25) coincides with the results of the paper [3].

Now we consider some generalizations of the extremal projector for the case of the quantum algebra $U_q(sl(2,\mathbb{C}))$.

12 'Adjoint extremal projectors' for $U_q(sl(2,\mathbf{C}))$

Let $\mathbf{J}^2$ be the Casimir invariant for $U_q(sl(2,\mathbf{C}))$:

$$\mathbf{J}^2 = \tfrac{1}{2}(J_+J_- + J_-J_+ + [2][J_0]^2)\ . \tag{12.1}$$

One easily verifies that:

$$\mathbf{J}^2 = X + [J_0][J_0+1]\ , \tag{12.2}$$

where we use the notation

$$X := J_-J_+\ . \tag{12.3}$$

It is evident that the first equation (9.11) for the extremal projector (9.9) can be rewritten in the form

$$Xp = pX = 0\ . \tag{12.4}$$

By using the Casimir operator $\mathbf{J}^2$ we can rewrite this equation as follows

$$\mathbf{J}^2p = p\mathbf{J}^2 = [J_0][J_0+1]p\ . \tag{12.5}$$

Thus the extremal projector p is an eigenvector for the Casimir operator $\mathbf{J}^2$ with the eigenvalue $[J_0][J_0+1]$. The equation (12.4) is similar to the algebraic equation for the δ-function:

$$x\delta(x) = 0\ . \tag{12.6}$$

Let us continue this analogy. It is known that $\delta(x+\epsilon)$ is the generating function for derivatives of δ–function, i.e.

$$\delta^{(n)}(x) = \frac{d^n\delta(x+\epsilon)}{d\epsilon^n}\Big|_{\epsilon=0}\ . \tag{12.7}$$

Introduce an element $p(\epsilon)$ of the form

$$p(\epsilon) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\overline{\Gamma}(n+1-\epsilon)\overline{\Gamma}(2J_0+2+\epsilon)} J_-^n J_+^n\ . \tag{12.8}$$

The element $p(\epsilon)$ is an analog of the generating function $\delta(x+\epsilon)$. Its properties are described in the proposition.

Proposition 12.1 *The element $p(\epsilon)$ satisfies the equation*

$$Xp(\epsilon) = p(\epsilon)X = [\epsilon][2J_0+1+\epsilon]p(\epsilon) \tag{12.9}$$

or

$$\mathbf{J}^2p(\epsilon) = p(\epsilon)\mathbf{J}^2 = [J_0+\epsilon][J_0+1+\epsilon]p(\epsilon)\ . \tag{12.10}$$

Proof. The equation (12.9) is easily verified by direct calculation. The equation (12.10) is a consequence of (12.9).

We see from eq. (12.10) that the element $p(\epsilon)$ is a eigenvector of the Casimir operator $\mathbf{J}^2$ with the eigenvalue $[J_0+\epsilon][J_0+\epsilon+1]$.

Let us introduce a scaling–derivative $\tilde{D}_x f(x)$ of a function $f(x)$ depending on a variable x as follows

$$\tilde{D}_x f(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{[\Delta x]} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(\ln q)^{-1} f_x^{'}(x))\ , \tag{12.11}$$

where f_x' is the usual derivative of the function $f(x)$.

Let $\tilde{p}^{(n)}$, $n=0,1,2,\ldots$, be scaling-derivatives of $p(\epsilon)$ at $\epsilon=0$, i.e.

$$\tilde{p}^{(n)} = (\tilde{D}_\epsilon)^n p(\epsilon)\ |_{\epsilon=0}\ . \tag{12.12}$$

Proposition 12.2 *The elements $\tilde{p}^{(n)}$, $n=0,1,2,\ldots$, satisfy the algebraic equations*

$$X\tilde{p}^{(n)} = \tilde{p}^{(n)}X = \sum_{l=0}^{n-1} a_l^n \tilde{p}^{(l)}\ , \qquad \text{for } n=0,1,2,\ldots\ , \tag{12.13}$$

where

$$a_l^n = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{n-l-2}(q^{\frac{2J_0+1}{2}} + (-1)^{n-l} q^{-\frac{2J_0+1}{2}}) \frac{n!}{l!(n-l)!}\ . \tag{12.14}$$

The elements $\tilde{p}^{(n)}$ can be redefined to $p^{(n)}$ such that they will satisfy the simple equations

$$Xp^{(n)} = p^{(n)}X = p^{(n-1)} \ , \qquad \text{for } \ n = 0, 2, 1, \ldots \tag{12.15}$$

or

$$\mathbf{J}^2 p^n = p^{(n)}\mathbf{J}^2 = [J_0][J_0 + 1]p^{(n)} + p^{(n-1)} \ , \qquad \text{for } \ n = 0, 1, 2, \ldots \ . \tag{12.16}$$

Proof. Applying the scaling–differentiation operator $(\tilde{D}_\epsilon)^n$ to eq. (12.9) and putting $\epsilon = 0$ we obtain the equations (12.13). Let

$$p^n = \sum_{l=0}^{n} d_l^n \tilde{p}^{(l)} \ . \tag{12.17}$$

Substituting this in eq. (12.15) we obtain the system of equations

$$\sum_{k=l+1}^{n} d_k^n a_l^k = d_l^{n-1} \ , \qquad \text{for } \ l = 0, 1, 2, \ldots, n-1 \ . \tag{12.18}$$

This system has a unique solution if $d_1^1 = [2J_0 + 1]^{-1}$ and $d_0^m = 0$ for $m = 1, 2, \ldots, n-1$. We shall not present the solution here, since it has a cumbersome form.

Remark. (i) The elements $p^{(n)}$, $n = 1, 2, \ldots$, are adjoint-vectors of the Casimir operator $\mathbf{J}^2$ with eigenvalue $[J_0][J_0 + 1]$. They are joined to the eigenvector p of $\mathbf{J}^2$. In this connection the element $p^{(n)}$ is called the 'adjoint extremal projector' of the n-th order.

(ii) The elements $(-1)^n (n!) p^{(n)}$, $n = 0, 1, 2, \ldots$, are analogs of the δ–function and its derivatives, since they satisfy the same algebraic equations.

(iii) In limit $q \to 1$ the elements $p^{(n)}(\tilde{p}^{(n)})$, $n = 0, 1, 2, \ldots$, turn into the corresponding elements of $sl(2, \mathbf{C})$ [27].

It was found that the 'adjoint extremal projectors', $p^{(n)}(U_q(sl(2))$, $n = 1, 2, \ldots$, are closely connected with a special class of decomposable representations for quantum algebra $U_q(sl(2))$ (details see [6]).

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